Borel Determinacy in 50 ($+\varepsilon$) Minutes

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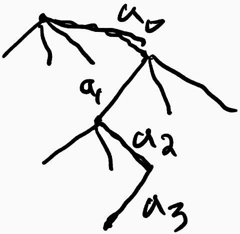
Games

I	a_0		a_2		
					$\dots$
II		a_1		a_3	

Gale-Stewart games; infinite two player games of perfect information where the players, denoted I and II, alternate moves.



Games

Given a nonempty set of moves M and a **payoff** set $A \subseteq M^{\mathbb{N}}$, we define the game $G(A)$:

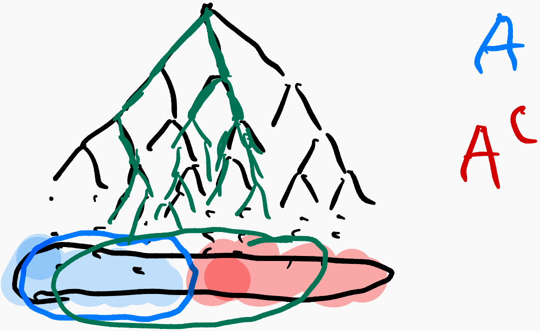
- A **position** is a finite sequence $p \in M^{<\mathbb{N}}$
- A **run** is an infinite sequence $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$
- Players I and II take turns playing **moves** $x \in M$
- Player I wins iff the run $(x_n)_{n\in\mathbb{N}} \in A$

Games with Rules

In practice we want to play a **game with rules**:

- Restrict moves to a subtree (without leaves), say T
- Equivalent to games without rules, up to changing the payoff set

$G(A; T)$:



Games: Determinacy

Definition

A game $G(A; T)$ is **determined** if one of the players has a winning strategy.

A **strategy** for player I is a function $\sigma : T \to M$ that tells the player what move to play at any even position $p \in T$ and is **winning** for I if every run consistent with σ is in A.

A strategy τ for II is defined analogously.

$\exists a_0 \forall a_1 \exists a_2 \forall a_3 \ldots \forall a_k (a_0, a_1, \ldots) \in A$

$\forall a_0 \exists a_1 \forall a \ldots \ldots \exists a_k (a_0, a_1, \ldots) \in A^c$

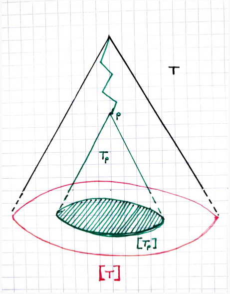
Rem: Bounded games are determined

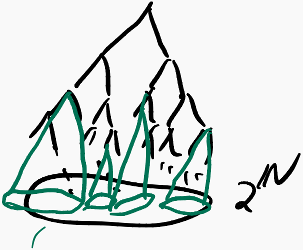
Games: Topology on the game tree

We equip $[T]$ with the topology whose basic open sets are $[T_p]$, $p \in T$, where

$$T_p := \{q \in T : (q \subseteq p) \vee (p \subseteq q)\}$$

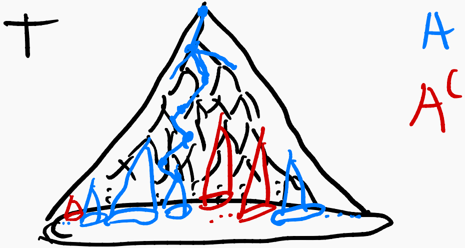
denotes the **game subtree** at position p.





Games: Clopen Determinacy

Clopen games = finite games



Theorem (Clopen Determinacy)

If $A \subseteq [T]$ is clopen, then $G(A; T)$ is determined.

Proof.

Suppose II doesn't have a winning strategy. Call a node "heavy" if II doesn't have a winning strategy from that point. Chase the heaviness into A. □

Previous Determinacy Results

Theorem (Gale, Stewart (1953))

Open/Closed sets (i.e. Π^0_1) are determined.

Theorem (Wolfe (1955), Davis (1964), Paris (1972))

Σ^0_2, *resp.* Π^0_3, *resp.* Σ^0_4 *sets are determined.*

⋮

⋮

Theorem (Borel Determinacy; Martin (1975))

All Borel sets are determined.

Motivation: Why do we care?

It turns out that regularity properties of subsets of a Polish space are naturally deduced from the determinacy of infinite games, including:

- measurability
- Baire measurability
- the perfect set property

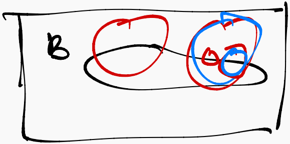
Borel determinacy tells us that Borel sets are the "nicest" possible.

Example: The Perfect Set Property

Let X be a perfect Polish space.

Definition

A set $B \subseteq X$ has the **perfect set property** (PSP) if it's either countable or contains a Cantor set ($2^{\mathbb{N}}$).



X

U ctbl

I $(U_0^0 \; U_1^0 \;)$

II $\quad 0$

$\bigcap U_n = \{x\}$

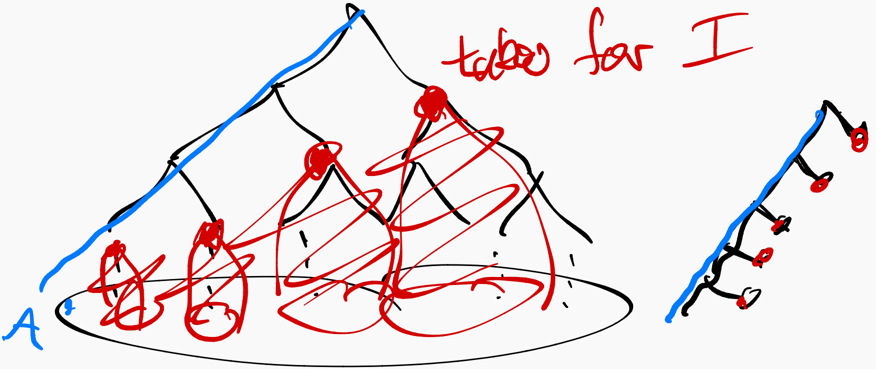
I wins if $x \in B$

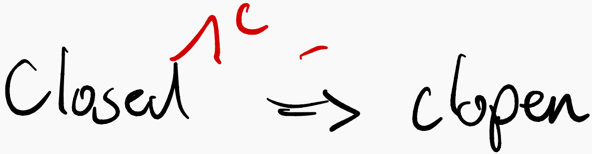
Thm. I has a winning strat $\iff 2^{\mathbb{N}} \subseteq B$

II has " " $\iff B$ ctbl

Proof of Borel Determinacy

... but first a slight detour into taboos



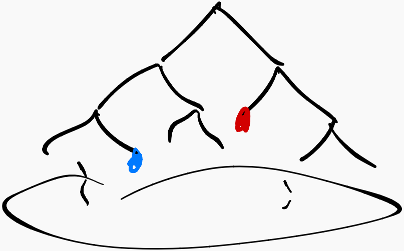


Games with taboos

Definition (Game tree with taboos)

A **game tree with taboos** is a triple $\mathbf{T} := \langle T, \mathcal{T}_{\mathrm{I}}, \mathcal{T}_{\mathrm{II}} \rangle$ where

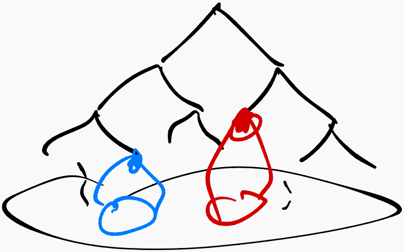
- T is game tree, but with leaves
- $\mathcal{T}_{\mathrm{I}}$ is the set of taboos for player I
- $\mathcal{T}_{\mathrm{II}}$ is the set of taboos for player II



Topology on games with taboos

For a game with taboos $G(A; \mathbf{T})$, we still consider only the space of infinite branches equipped with the topology as before.

Games with taboos can be modeled as infinite games without taboos:



Remark: This may change the Borel complexity of subsets of $[T]$

Determinacy of games with taboos

Lemma

Clopen games with taboos are determined.

Note: Clopen determinacy for games without taboos does not give us this result for free! (but the proof is similar in spirit)

Idea of Proof of Borel Determinacy

Given a Borel game $G(A; \mathbf{T})$ we want to build an auxiliary clopen game $G(\tilde{A}, \tilde{\mathbf{T}})$ s.t. winning strategies in $G(\tilde{A}, \tilde{\mathbf{T}})$ map to winning strategies in $G(A; \mathbf{T})$.

Game coverings

Definition (Covering)

A covering of a game tree $\mathbf{T}$ is a tuple $\langle \tilde{\mathbf{T}}, \pi, \phi \rangle$, of a game tree $\tilde{\mathbf{T}}$, a position map $\pi : \tilde{\mathbf{T}} \Rightarrow \mathbf{T}$, and a strategy map $\phi : \tilde{\mathbf{T}} \xRightarrow{\mathcal{S}} \mathbf{T}$, such that:

Lemma

For all $A \subseteq [T]$, if $\tilde{\sigma}$ is a winning strategy for $G(\pi^{-1}(A); \tilde{\mathbf{T}})$, then $\sigma := \phi(\tilde{\sigma})$ is a winning strategy for $G(A; \mathbf{T})$.

$c : \tilde{T} \Rightarrow T$

Coverings: unraveling

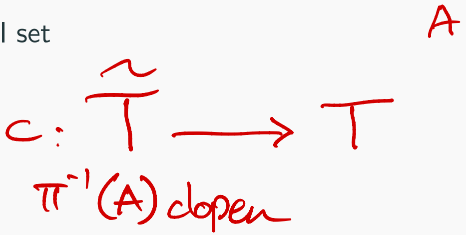
Definition (Unraveling)

Given a set $A \subseteq [T]$, we say a covering $\langle \tilde{\mathbf{T}}, \pi, \phi \rangle$ of $\mathbf{T}$ **unravels** A if $\pi^{-1}(A)$ is clopen in $[\tilde{T}]$.

Corollary

If there is a covering of $\mathbf{T}$ *that unravels* $A \subseteq [T]$*, then* $G(A; \mathbf{T})$ *is determined.*

Our goal: Unravel every Borel set

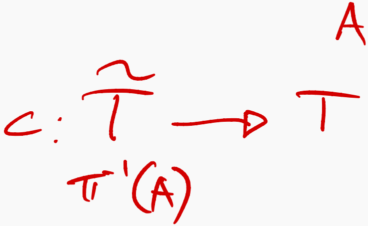


Proof by Induction

Recall each Borel set is obtained from open (or closed) sets by applying complements and ctbl unions.

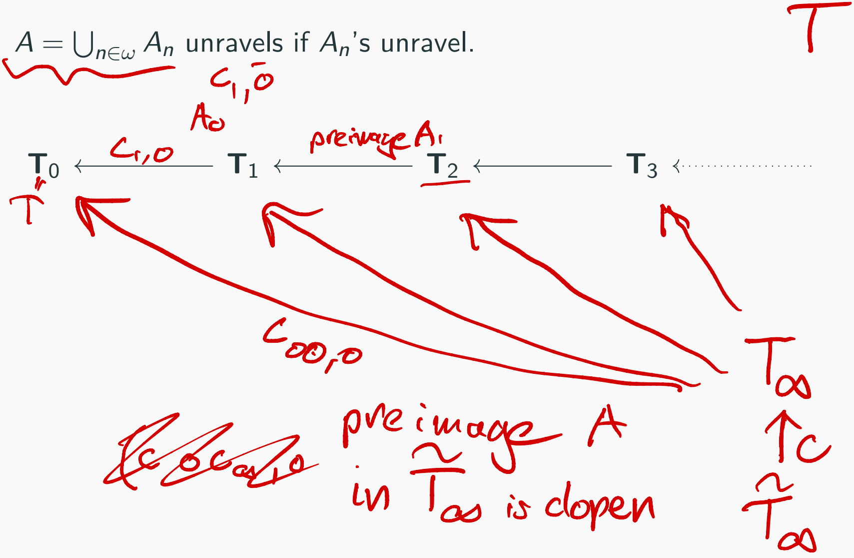
Inductive proof:

- Base case: unravel closed sets
- Complements: A unraveled $\implies A^c$ unraveled
- Ctbl $\bigcup$: each A_n unraveled $\implies \bigcup_{n\in\omega} A_n$ unraveled

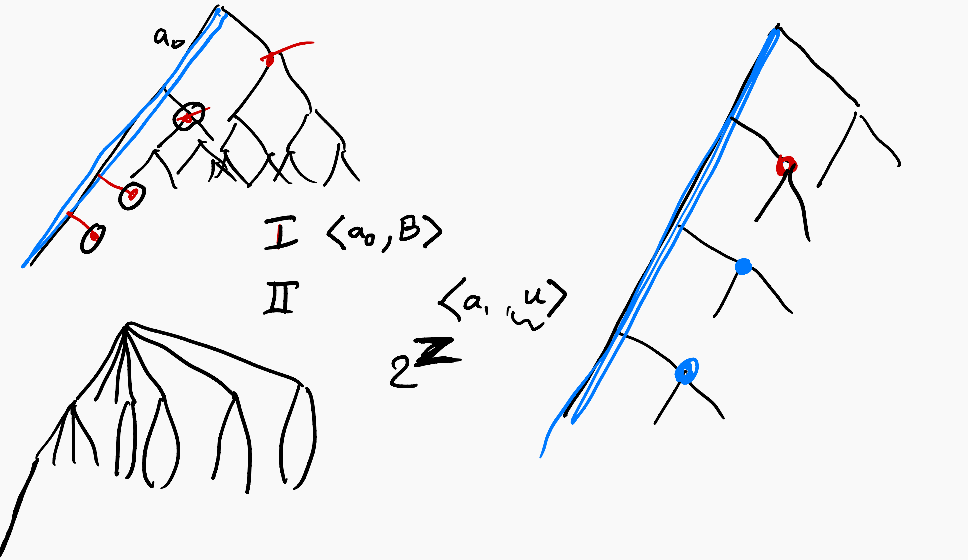


$A = \bigcup_{n \in \omega} A_n$ unravels if A_n's unravel.

$$\mathbf{T}_0 \leftarrow \mathbf{T}_1 \leftarrow \mathbf{T}_2 \leftarrow \mathbf{T}_3 \leftarrow \cdots$$



A word on the base case



Questions?

Thank you!

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